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2A

1. Let $f(x)=\sqrt{a+bx}$.

$$f(x) \approx f(0)+f'(0)(x-0)$$
$$= \sqrt{a}+\frac{b}{2\sqrt{a}}x$$

$$f'(x)=\frac{1}{2}\cdot\frac{1}{\sqrt{a+bx}}\cdot b$$
$$=\frac{b}{2\sqrt{a+bx}}$$

3. Let $f(x)=\dfrac{(1+x)^{3/2}}{1+2x}$.

$$f'(x)=\frac{\frac{3}{2}\sqrt{1+x}\,(1+2x)-(1+x)^{3/2}\cdot 2}{(1+2x)^2}$$
$$=\frac{\sqrt{1+x}\left(\frac{3}{2}(1+2x)-2-2x\right)}{(1+2x)^2}$$
$$=\frac{\sqrt{1+x}\left(-\frac{1}{2}+x\right)}{(1+2x)^2}$$

$$f(x)\approx f(0)+f'(0)(x-0)$$
$$=1+\frac{1\left(-\frac{1}{2}\right)}{1}x$$
$$=1-\frac{x}{2}$$

6. Let $f(\theta)=\tan\theta$.

$$f(\theta)\approx f(0)+f'(0)(\theta-0)+\frac{f''(0)}{2}(\theta-0)^2 \text{ when } \theta\approx 0.$$

$$f(\theta) \approx 0 + \sec^2(0)\,\theta + \frac{2\tan(0)\sec^2(0)}{2}\theta^2$$
$$= \theta$$

$$\frac{d^2}{dx^2}(\tan\theta)$$
$$= \frac{d}{dx}\sec^2\theta$$
$$= \frac{d}{dx}(\cos\theta)^{-2}$$
$$= -2\sec^3\theta(-\sin\theta)$$
$$= 2\tan\theta\sec^2\theta$$

11. $pv^k = C$

Let $f(V) = p. \Rightarrow f(V) = \frac{C}{V^k}$

$$f'(V) = \frac{-kC}{V^{k+1}}$$
$$f''(V) = \frac{k(k+1)C}{V^{k+2}}$$

$$f(V_0 + \vec{D}v) \approx f(V_0) + f'(V_0)\left((V_0+\vec{D}v) - V_0\right) + \frac{f''(V_0)}{2}\left((V_0+\vec{D}v) - V_0\right)^2 \quad \text{as } \vec{D}v \approx 0$$
$$= \frac{C}{V_0^k} + \frac{-kC}{V_0^{k+1}}(\vec{D}v) + \frac{k(k+1)C}{2V_0^{k+2}}\vec{D}v^2$$

$-(1-x)^{-2}$

12. a) $f(x) = \frac{e^x}{1-x}$ (quadratic, $x \approx 0$)

$g = e^x,\ h = (1-x)^{-1}$

Using $Q(gh) = Q(Q(g)Q(h))$.

$$Q(g) = 1 + x + \frac{x^2}{2}$$
$$Q(h) = 1 + (-1)(1-0)^{-2}(-1)x + \frac{(-2)(1-0)^{-3}(-1)}{2}x^2$$
$$= 1 + x + x^2$$

$$f(x) \approx Q\left(\left(1 + x + \frac{x^2}{2}\right)\left(1 + x + x^2\right)\right)$$
$$= 1 + (x + x) + \left(\frac{x^2}{2} + x^2 + x^2\right)$$
$$= 1 + 2x + \frac{5}{2}x^2$$

d) $f(x) = \ln\cos x$

$$f'(x) = \frac{1}{\cos x}(-\sin x)$$
$$= -\tan x$$
$$f''(x) = -\sec^2 x$$

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$
$$= 0 + (0)x + \frac{-1}{2}x^2$$
$$= -\frac{x^2}{2}$$

e) $f(x) = x\ln x$ (quadratic, $x \approx 1$)

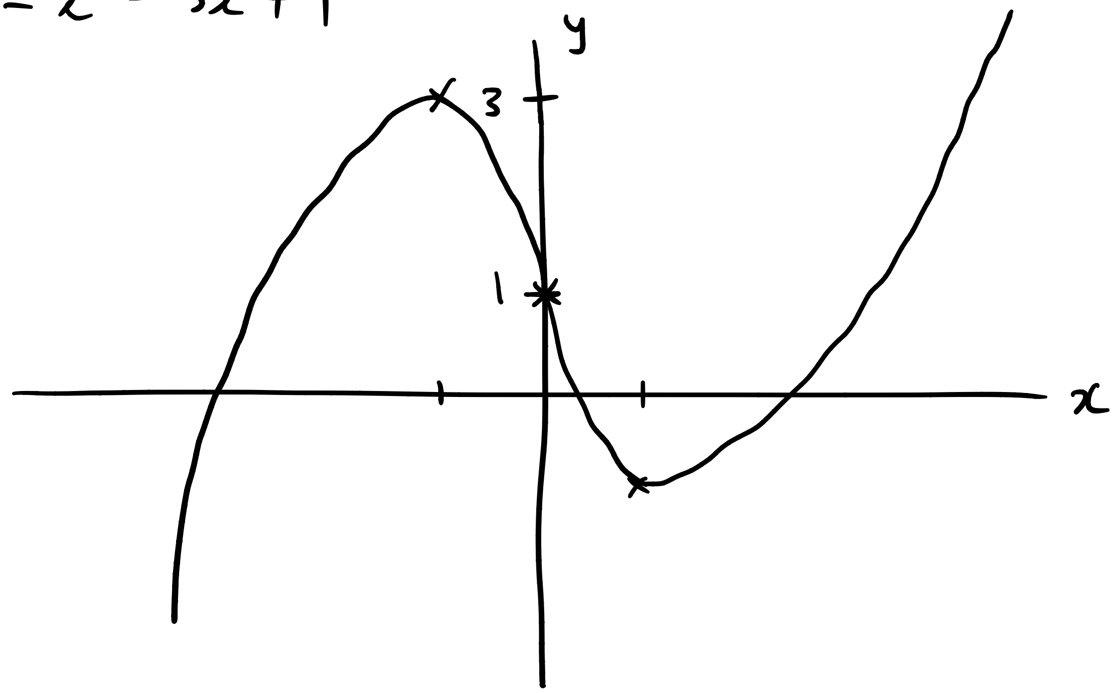
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$
$$= f(1) + (1 + \ln 1)(x - 1) + \frac{1/1}{2}(x-1)^2$$
$$= 0 + x - 1 + \frac{1}{2}(x^2 - 2x + 1)$$
$$= \frac{x^2}{2} - \frac{1}{2}$$
$$= \frac{1}{2}(x+1)(x-1)$$

$$f'(x) = \ln x + x\frac{1}{x}$$
$$= 1 + \ln x$$
$$f''(x) = \frac{1}{x}$$

2B

1.

a) $y = x^3 - 3x + 1$



$y' = 3x^2 - 3$

$= 3(x+1)(x-1)$

$\therefore x = \pm 1$ are critical points

$\Rightarrow y' > 0$ when $x > 1$ and $x < -1$.

y is increasing.

$y' < 0$ when $-1 < x < 1$.

y is decreasing.

As $x \to \pm\infty$, $y \to x^3$.

$\Rightarrow y \to \infty$ as $x \to \infty$,

$y \to -\infty$ as $x \to -\infty$

$y'' = 6x$

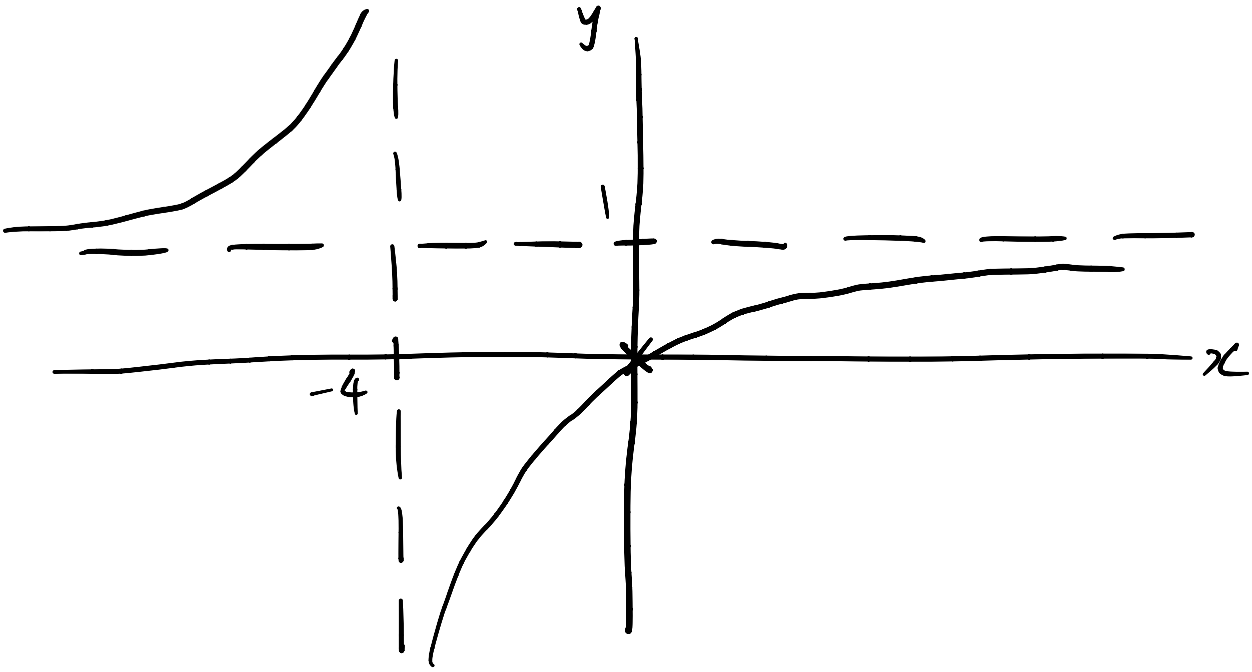
$\Rightarrow y'' > 0$ when $x > 0$, y is concave up.

$y'' < 0$ when $x < 0$, y is concave down.

$\therefore y$ crosses the x-axis 3 times.

2. $y'' = 0$ at $x = 0$. $(0, 0)$ is the inflection point.

e) $y = \dfrac{x}{x+4} = \dfrac{1}{1+\frac{4}{x}}$



$\lim\limits_{x \to \pm\infty} \dfrac{1}{1+\frac{4}{x}} = 1$

$y' = \dfrac{1(x+4) - x(1)}{(x+4)^2}$

$= \dfrac{4}{(x+4)^2}$

$\therefore$ $y' \neq 0$, $y' > 0$.

$\Rightarrow$ y is always increasing.

$\lim\limits_{x \to -4^+} \dfrac{x}{x+4}$

$= \dfrac{(-4)^+}{(-4)^+ + 4}$

$= -\infty$

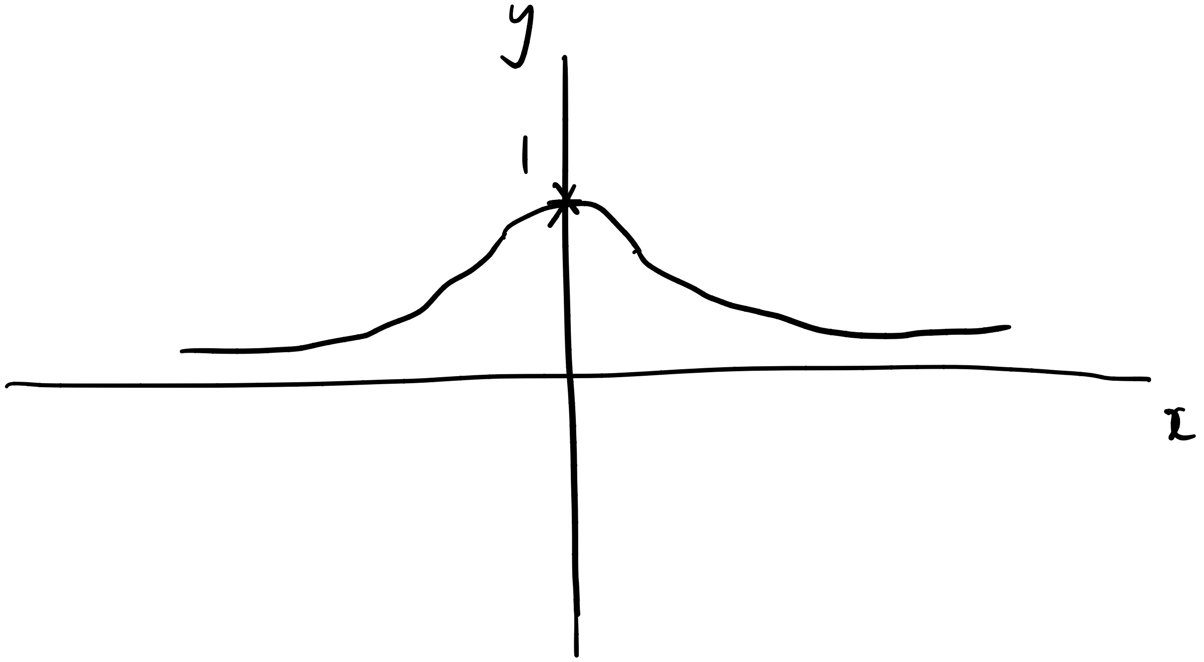
$\lim\limits_{x \to -4^-} \dfrac{x}{x+4} = \infty$

$\therefore$ y is increasing $(-\infty, -4)$, $(-4, \infty)$.

Only 1 solution to $y=0$ at $x=0$.

2. $y'' = -\dfrac{8}{(x+4)^3}$ $\therefore$ no inflection point.

h) $y = e^{-x^2}$



$y' = -2xe^{-x^2}$

$y' = 0$

$\Rightarrow -2x = 0$

$x = 0$

$\Rightarrow y' > 0$, y is increasing when $x < 0$.

$y' < 0$, y is decreasing when $x > 0$.

As $x \to \pm\infty$, $y \to 0$.

$\therefore$ y is increasing at $(-\infty, 0)$ and decreasing at $(0, \infty)$.

No solutions at $y=0$.

2. $y'' = -2e^{-x^2} + (-2x) - 2xe^{-x^2} \Rightarrow 1 - 2x^2 = 0$

$= -2e^{-x^2}(1 - 2x^2)$

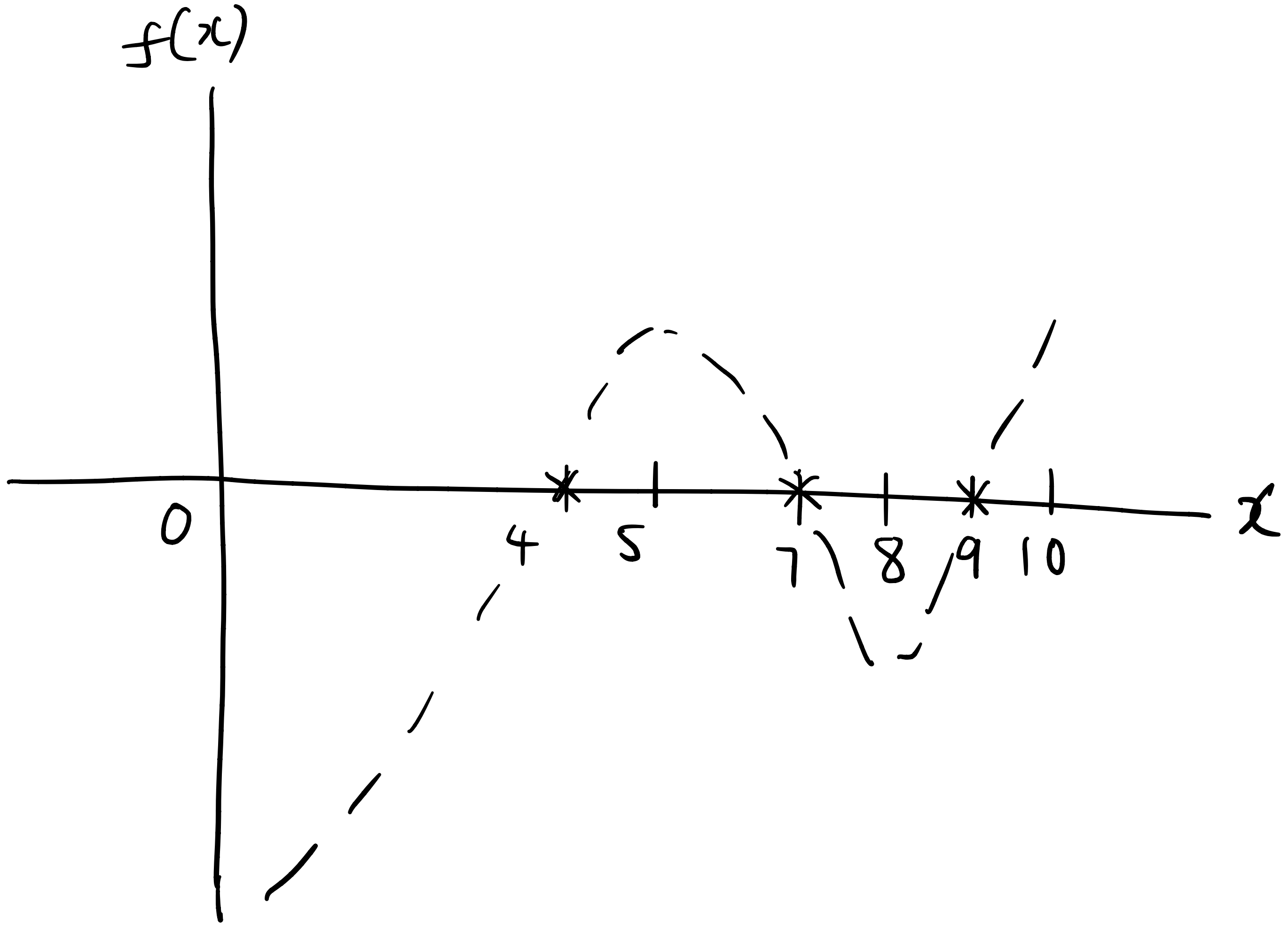
$\therefore x = \pm \frac{\sqrt{2}}{2}$ are the inflection points.

4. $f(x) = 0$ at $x = 4, 7, 9$.

$f'(x) > 0$ on $0 < x < 5 \Rightarrow f(x)$ is increasing

$f'(x) < 0$ on $5 < x < 8 \Rightarrow f(x)$ is decreasing

$f(x) = \{f(x) : 0 \leq x \leq 10\}$



$\therefore$ No, just that at $x = 5$, a maximum value is obtained and at $x = 8$, a minimum value is obtained.

6.

a) $f'(x) = 0$ at $x = -1, 1$

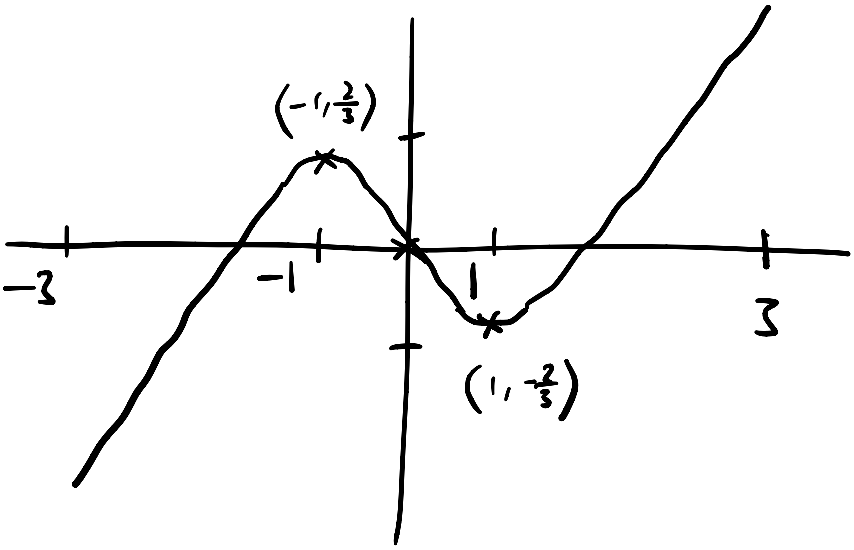
$$(x+1)(x-1) = 0$$

$$\Rightarrow f'(x) = x^2 - 1$$

$$\Rightarrow f(x) = \frac{x^3}{3} - x + C$$

Let $C = 0$, $f(x) = \frac{x^3}{3} - x$

b)

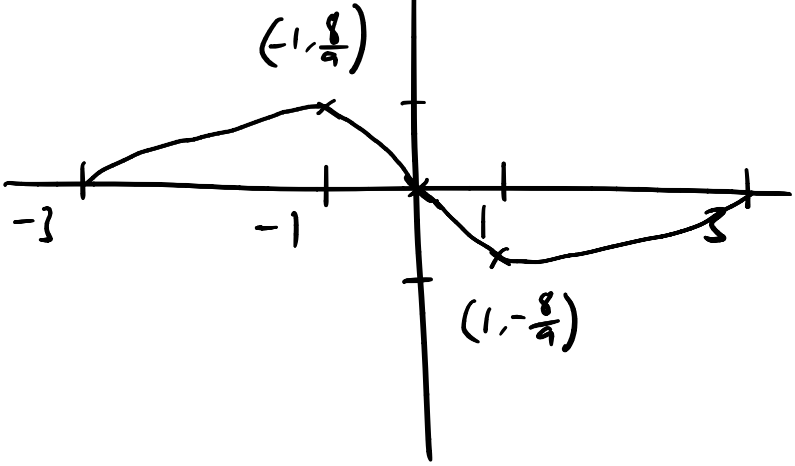


$f(x)$ is increasing when $x > 1$ and $x < -1$,
decreasing when $-1 < x < 1$

$f(x) \to \infty$ as $x \to \infty$

$f(x) \to -\infty$ as $x \to -\infty$

c)



$$f(x) = \frac{x^3}{9} - x$$

$$f'(x) = \frac{1}{3}x^2 - 1$$

7.

a) $f(x)$ is increasing and $f'(a)$ exists.

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$f(x)$ is increasing $\Rightarrow$ $\Delta y > 0 \Rightarrow \Delta x > 0$
$\Delta y < 0 \Rightarrow \Delta y < 0$

In both cases, $\frac{\Delta y}{\Delta x} > 0$.

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a) \geq 0.$$

b) A function $\frac{\Delta y}{\Delta x} > 0$ does not mean $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} > 0$.

$f(x) = x^3$, $f'(0) = 3(0)^2 = 0$.